On the stability of a heterogeneous shear layer subject to a body force

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The effects of density variation and body force on the stability of a heterogeneous horizontal shear layer are investigated. The density is assumed to decrease exponentially with height, and the body force is assumed to be derivable from a potential; the velocity distribution in the shear layer is taken to be $U(y) = \tanh y$. The method of small disturbances is employed to obtain a family of neutral stability curves depending on the choice of the Richardson number. It is demonstrated, furthermore, that the value of the critical Richardson number depends on the magnitude of the non-dimensional density gradient.

1. Introduction

It is a frequent occurrence in nature that two fluids of different densities flow one on top of the other. If the flow is predominantly horizontal, and if the density diminishes rapidly in the upward direction, then the process of turbulent mixing must cause heavier fluid elements to be moved above lighter ones and lighter fluid elements below heavier ones. Both displacements consume energy that has to be extracted from the mean flow at the expense of energy that might be available for the maintenance of turbulence.† The same considerations apply quite generally to work against any body force. The present paper, an extension of an earlier analysis (Menkes 1959), is one of a series of researches (see, for example, Prandtl 1931; Taylor 1931; Goldstein 1931; Drazin 1958) undertaken to establish the limits of stability of a shear flow in a stably stratified medium.

2. Analysis

The equations of motion governing the behaviour of an incompressible inviscid fluid under the action of a body force, $g\nabla y$, are Euler's equation,

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} - g\nabla y, \tag{1}$$

the condition of incompressibility,

$$\frac{D\rho}{Dt} = 0, \tag{2}$$

† This argument, which follows Prandtl's exposition (Prandtl, 1952, p. 131), presupposes that the total kinetic energy can be resolved into two terms: one term represents the contribution of the mean flow and the other that giving rise to the turbulent Reynolds stresses. The energy partition is assumed to be unaffected by the density stratification.

and the equation of continuity,

$$\nabla . \mathbf{u} = 0. \tag{3}$$

The velocity components $\mathbf{u}$, the pressure p, and the density ρ are assumed to consist of a time-independent part and of a perturbation. Thus we write

$$u = V[U(y) + \psi'_y(x,y,t)], \quad v = -V\psi'_x(x,y,t), \tag{4a}$$

$$p = P(y) + p'(x,y,t), \tag{4b}$$

$$\rho = \rho_0[\bar{\rho}(y) + \rho'(x,y,t)], \tag{4c}$$

$$\psi' = \psi(y)\exp[ik(x-ct)], \quad (c = c_r + ic_i). \tag{4d}$$

Here V is the velocity at $y = \pm\infty$ and ψ' is a perturbation stream function; $U(y)$, $P(y)$, and $\bar{\rho}(y)$ describe the ambient state whose stability is to be investigated; k is a wave-number and c a complex phase velocity. Also, $x = x_1/d$ and $y = y_1/d$, where x_1 and y_1 are the physical co-ordinates and d is so chosen that $dU/dy = 1$ at $y = 0$; thus d characterizes the width of the transition layer. Denoting the Froude number, V^2/gd, by F, setting $\bar{\rho} = \exp(-2Ly)$, where L is a dimensionless density gradient, and eliminating ρ' and p' from equations (1), (2), and (3) yields

$$(U-c)(\psi'' - k^2\psi) - U''\psi + (\ln\bar{\rho})'[(U-c)\psi' - (U-c)'\psi] - \frac{(\ln\bar{\rho})'}{F}\frac{\psi}{U-c} = 0, \tag{5}$$

where primes denote differentiation with respect to y.

At this point we introduce the primary velocity distribution $U(y) = \tanh y$ as the independent variable. Denoting now by primes differentiation with respect to U, and setting $\psi(y) = \phi(U)$ and $2L/F = J$, we derive the equation

$$\phi'' + a(U)\phi' + b(U)\phi = 0, \tag{6}$$

where

$$a(U) = \frac{2(U+L)}{(U+1)(U-1)}, \tag{7a}$$

$$b(U) = \frac{J}{(U-c)^2(U+1)^2(U-1)^2} - \frac{k^2}{(U+1)^2(U-1)^2} - \frac{2(U+L)}{(U-c)(U+1)(U-1)}, \tag{7b}$$

with the boundary conditions $k\phi(U) = 0$, at $U = \pm 1$. (J is the Richardson number.)

Equation (6) is of a rather simple type. Its singularities, which are located at ± 1 and c, are regular singularities. It can be demonstrated that the point at infinity is also a regular singularity. The substitution into equation (6) of

$$Z = (U-1)^{-\alpha_1}(U-c)^{-\alpha_2}(U+1)^{-\alpha_3}\phi$$

yields an equation that has at least one bounded solution at each of the singularities. The α_i are defined by

$$\alpha_1 = \frac{L}{2}\left\{\left[1 + \frac{k^2}{L^2} - \frac{2J}{L^2(1-c)^2}\right]^{\frac{1}{2}} - 1\right\}, \quad \alpha_2 = \frac{1}{2}\left\{1 - \left[1 - \frac{8J}{(1-c^2)^2}\right]^{\frac{1}{2}}\right\},$$

$$\alpha_3 = \frac{L}{2}\left\{\left[1 + \frac{k^2}{L^2} - \frac{2J}{L^2(1+c)^2}\right]^{\frac{1}{2}} - 1\right\},$$

and each represents one of the indices relative to the finite points of the singularity. After this transformation, equation (6) reads

$$Z''+\left[\frac{1-\alpha_1}{U-1}+\frac{1-\alpha_2}{U-c}+\frac{1-\alpha_3}{U+1}\right]Z'+\frac{(\sigma\tau U-r)}{(U-1)(U-c)(U+1)}Z=0, \tag{8}$$

with the boundary conditions replaced by regularity conditions on $Z(U)$ at $U=\pm 1$ and c. A sufficient condition for the existence of a trivial solution $Z=$ constant (which, of course, satisfies the boundary conditions identically) is given formally by

$$\sigma\tau=0, \tag{9}$$

$$r=0, \tag{10}$$

where σ and τ are the indices relative to the point at infinity and r is an accessory parameter (see Bieberbach 1953). After a considerable amount of tedious, but essentially straightforward, algebraic manipulation, we obtain from equations (9) and (10) the explicit relations:

$$L=-c\left[1-\frac{4J}{(1-c^2)^2}\right]-\tfrac{1}{4}(1-\mu)\left[R(1-c)(1-\xi)^{\frac{1}{2}}-R(1+c)(1-\eta)^{\frac{1}{2}}-2c\right], \tag{11}$$

and

$$R^2=\frac{R(\mu-2)[(1-\xi)^{\frac{1}{2}}(1-\eta)^{\frac{1}{2}}]+J(1-c^2)^{-2}+2(\mu+1)}{1+[(1-\xi)(1-\eta)]^{\frac{1}{2}}}, \tag{12}$$

where

$$R^2=L^2+k^2,\quad \xi=\frac{2J}{R^2(1-c)^2},\quad \eta=\frac{2J}{R^2(1+c)^2},\quad \mu=\left[1-\frac{8J}{(1-c^2)^2}\right]^{\frac{1}{2}}.$$

It is to be noted that equations (11) and (12) are not homogeneous in any of the quantities J, L, k, and c; however, there are two equations in four unknowns. It was found convenient from a computational point of view to consider L and R^2 as the primary variables, and J and c as parameters. The equations were solved numerically by an iteration technique on an IBM 704 digital computer.

3. Discussion and results

In the usual case the neutral stability curve represents possible neutral disturbances and separates the stable from the unstable ones, with no 'forbidden' disturbances present anywhere. The stability boundaries displayed in figure 1 have a meaning slightly different from that commonly accepted. A particular boundary separates unstable disturbances from stable ones *and* from those that are physically not realizable. This may be stated in a different way: as the boundary is approached from the inside along an arbitrary path one passes over possible disturbances, and the closer one gets to the boundary the smaller will be the amplification. On the other hand, when the boundary is approached from the outside one cannot be sure whether an arbitrary path consists only of permitted disturbances or not. This implies that the only statement that can be made about the attenuation is that if the path consists only of a succession of possible disturbances, then as the boundary is approached the attenuation decreases, to vanish at the boundary itself. For practical purposes, however, this distinction is immaterial since one may state without ambiguity the maximum value of k^2+L^2 that corresponds to an unstable disturbance, for a prescribed value of J.

The present analysis throws into relief the dependence of the stability on the relative magnitudes of the wave-number k and the parameter L. The governing quantity appears to be $k^2 + L^2$. The analysis of Taylor (1931) in which the parameter L was neglected indicated that as the wave-number decreased, the instability became more pronounced. By including the effect of the density gradient, one can stabilize a disturbance that was shown, in the absence of a density gradient, to be unstable by virtue of its small wave-number. The stabilizing effect of the density gradient is thus clearly demonstrated.

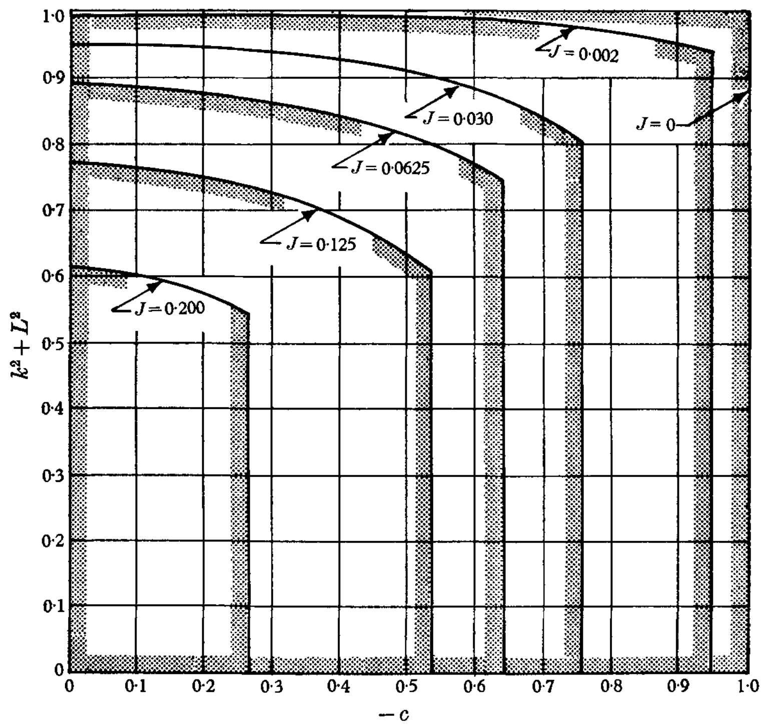

FIGURE 1. Stability map. The curves represent the boundaries between unstable disturbances (inside, shaded) and disturbances which are either damped or physically unrealizable.

The relationship that must exist between L and c, for given values of J, to obtain a neutral disturbance is displayed in figure 2. It is to be noted that the curves terminate before reaching the c-axis, thus implying that the only solution corresponding to $L = 0$ is $c = 0$, which is the one found by Drazin.

The accepted convention of referring to a critical Richardson number is unfortunate because it conjures up similarities with the critical Reynolds number of hydrodynamic stability theory to which it bears hardly any semblance. The critical Richardson number represents a number beyond which no virtual displacement† of the flow field appears possible if one forces the solution to be

† The displacement is taken to apply not only to spatial displacements but also to velocity, pressure, etc.

exponential in x as in t, and furthermore if one rules out the improper eigenfunctions associated with the continuous spectrum as demonstrated by Eliassen, Høiland & Riis (1953) and Case (1960).† In this sense, then, the flow is stable, so to say, by default. This is certainly an odd result, but one that has been obtained consistently by Taylor, Goldstein, Drazin, and the present author.

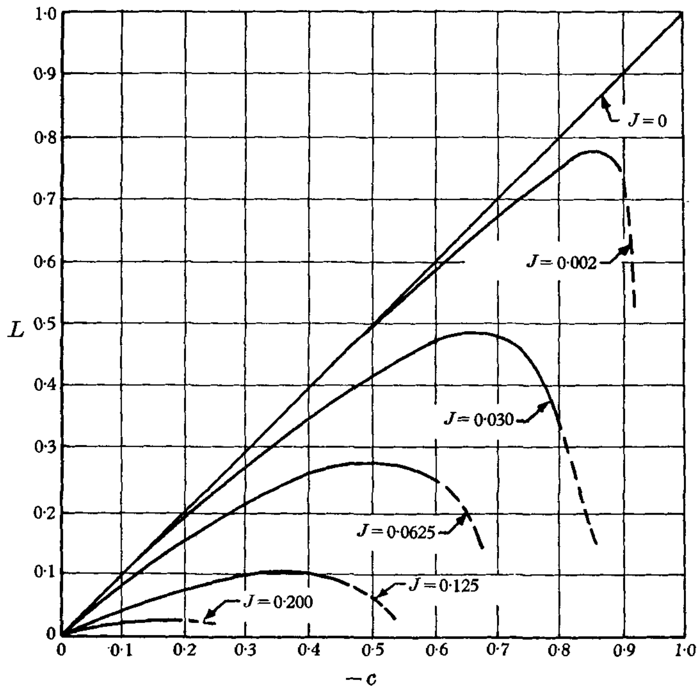

FIGURE 2. Neutral stability boundaries.

It appears to be a property of inviscid stability problems. No physical explanation can be offered, and the mathematical one provides little solace; it only states that, for $J > J_{cr}$, a physical quantity becomes imaginary when subjected to a virtual displacement (see Appendix).

The fact that the disturbance spectrum is finite, due to the presence of a cut-off wave-number, does not necessarily imply, however, that the asymptotic flow field is undetermined. It has been demonstrated by Case that, in general, the solutions associated with the continuous spectrum decay like $1/t$, while Carrier & Chang (1959) have shown that the inclusion of the continuous spectrum does not change the value of the cut-off wave-number. Thus, without solving the initial value problem in the present case, one may reason that the missing solutions will not affect the stability of the flow.

† The author is indebted to a referee for the precise formulation of the statement.

Appendix. The relationship between 'completeness' and the cut-off wave-number

The possibility that wave-like solutions to the disturbance equation may not exist can be anticipated from the following considerations. Any second-order differential equation of the type

$$Z''+g(U)Z'+h(U)Z = 0$$

can be reduced to the canonical form

$$W''+I(U)W = 0.$$

The invariant $I(U)$ is given by

$$I(U) = h-\tfrac{1}{4}g^2-\tfrac{1}{2}g',$$

while the dependent variable is given by $W = Z\exp\frac{1}{2}\int g(U)\,dU$. When this transformation is applied to equation (8), subject to the conditions (9) and (10), we find that the invariant is given by

$$I = \frac{1}{4}\left[\frac{1-\alpha_1}{(U-1)^2}+\frac{1-\alpha_2}{(U-c)^2}+\frac{1-\alpha_3}{(U+1)^2}\right]$$
$$-\frac{1}{2}\left[\frac{(1-\alpha_1)(1-\alpha_2)}{(U-1)(U-c)}+\frac{(1-\alpha_1)(1-\alpha_3)}{(U-1)(U+1)}+\frac{(1-\alpha_2)(1-\alpha_3)}{(U-c)(U+1)}\right].$$

For the case where I is constant, it is well known that $I > 0$ corresponds to wave-like solutions and that $I < 0$ corresponds to exponential solutions. The same criterion still holds if I is a function of the independent variable (see Mott 1952, chapter 1). The form of $I(U)$ as displayed above makes it plausible that for a certain combination of values of α_i, which are functions of L, J, k and c, the magnitude of $I(U)$ will become negative. If we now try to force the solution to be wave-like, we shall find that the velocity perturbation will turn out to be a purely imaginary quantity, indicating the physical impossibility of realizing such a solution. The detailed calculations have shown that this does actually take place when $J > (k^2+L^2)(1-c)^2$, or $J > \frac{1}{4}(1-c^2)^2$, whichever is smaller. The smaller value of the Richardson number is referred to as the critical one. This simple relationship between the invariant and the cut-off wave-number has apparently not been noticed before.

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